MATH 442: Mathematical Modeling

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Project Description: Phytoplankton modeling

Instructions: If you choose to work on this project, you may pick a partner or two to make a group of 2 or 3 to work together. You are to use pair programming techniques to do the MATLAB. You are to split the writing tasks and critique each others work, bringing everything together into one document at the end. Keep track of who does what and how each person contributes to this assignment; this will also be handed in. In addition to a writeup, students working on this project are expected to give a presentation of it.

Things to check before handing in the assignment:

1. Have you done everything you were asked to do/are able to do?
2. Is the math right? If not, get it fixed.
3. Is what you are trying to explain clear?
4. Do your graphs clearly show the conclusions you reach?
5. Are the spelling, grammar, and English usage correct and concise?
6. Do you think this is A, B or C level work?
7. Run the document through a spell checker.

By the end of this assignment, you and your partners should have a thorough understanding of the topics, the programming, and excellent explanations and examples to hand in – hopefully all A work.

Phytoplankton population growth[1]

Single nutrient, single species

This mathematical model is for phytoplankton population growth where there is a limited nutrient to support that growth.

The population is governed by the differential equations

$$\frac{dN_i}{dt} = \mu_i N_i - \nu N_i \tag{1}$$

$$\frac{dS_j}{dt} = \nu(\tilde{S}_j - S_j) - \sum_{i=1}^{n} Q_{ij}\mu_i N_i \tag{2}$$

This represents conditions of a chemostat set up where volume of the growth chamber is fixed, so inflow equals outflow; and a multiple substrates (nutrients) supplied in the source water limit the population growth of the phytoplankton species. You can find pictures of a chemostat setup online.

The indices i, j in the equations indicate that there can be more than one species and more than one nutrient, all of which interact in the chemostat.

The change in population density of the phytoplankton species can be simulated using equation 1 (units are cells liter^{-1} d^{-1}), where N_i is the population density of species i (cells liter^{-1}), μ_i is the specific growth rate of the population (d^{-1}), and ν is the hydraulic flushing rate (d^{-1}), which is just the inflow (liters d^{-1}) divided by the system volume (liters).

The change in the concentration of the growth-limiting substrate can be simulated using equation 2 (units of μmole-S liter^{-1} d^{-1}), where $\tilde{S}$ is the fixed concentration of the source substrate (μmole-S liter^{-1}), Q_{ij} is the fixed cellular content of S_j for the phytoplankton population i (μmole-S cell^{-1}), and other parameters are the same as previously defined.

The differential equations are linked through a mathematical equation, the Monod relationship, that describes the specific growth rate as a function of the availability of the growth-limiting substrate, where $\tilde{\mu}$ is the maximum specific growth rate for the population and k_{ij} is the half-saturation coefficient for substrate-limited population growth. These constants are species specific and can be determinied experimentally. This equation gives the dependence of the per-capita reproductive rate (μ_i) on the external concentration of the limiting nutrient (S_j):

$$\mu_i = \tilde{\mu}_i \min_j \left(\frac{S_j}{S_j + k_{ij}} \right)$$

[1] Prof. Dan Roelke from TAMU worked out the main details of this project. The work is his, but any mistakes in the description are mine.

Tasks

In order to get started, set up a meeting with me. You will need to learn a bit of theory which can be heavy going; it is strongly recommended that you talk to me about the project, then read sections 3.2, 3.5, 3.8 and 3.9 of Plankton Biology, which is available in a PDF on eLearning with the file title **SommerU_1989**) [Sommer(1989)]. If he is available, you may want to set up a meeting with Prof. Roelke to discuss the theory.

When you understand the basics of what is going on, start in on this task list. Generally students should be able to get at least into the third project step before the end of the semester, and demonstrate coexistence of two species with two nutrients. Keep me updated weekly on your progress; you will likely find you cannot undertake all of these tasks; the only way to make sure my expectations are in accordance with yours is to **communicate**.

1. Basics.
   - (a) Implement the model using one species and one substrate (nutrient). Try using these parameters: $\tilde{\mu} = 1.5$, $k = 0.5$, $Q = 50 \times 10^{-9}$, $\tilde{S} = 10$, $\nu = 0.5$, with initial conditions $N(0) = 1000$, $S(0) = \tilde{S}$.
   - (b) Perform a sensitivity analysis (systematically change the input parameters and see how it affects the output) to see how variations in $\tilde{\mu}$, k and Q influence the steady-state accumulated biomass, and the amount of time it takes the model to reach steady-state.
   - (c) Perform a sensitivity analysis to see how variations of ν and $\tilde{S}$ influence the steady-state accumulated biomass, and the amount of time it takes the model to reach steady-state.

2. Resource Saturation-Limitation Theory (see Figure 3.16 Panel B from [Sommer(1989)]).

   (a) Implement the model with three species and one substrate. Try using:
   $\tilde{\mu} = \begin{bmatrix} 1.5 \\ 2 \\ 1 \end{bmatrix}$,
   $k = \begin{bmatrix} 1.25 \\ 2.5 \\ 0.5 \end{bmatrix}$,
   $Q = \begin{bmatrix} 50 \times 10^{-9} \\ 50 \times 10^{-9} \\ 50 \times 10^{-9} \end{bmatrix}$,
   $\tilde{S} = 10$,
   $\nu = 0.5$,
   $N(0) = N_0 = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}$,
   $S(0) = S_0 = \tilde{S}$.

   (b) Which species wins the competition at prolonged conditions of substrate limitation? Could you have predicted this without running the model?

   (c) What does the succession pattern look like and why does it occur this way? Could you have predicted this without running the model?

   (d) using the Monod relationship, determine the substrate concentrations at which coexistence is possible.

3. Optimal resource ratio (Figure 3.5 and 3.16 panel A in [Sommer(1989)]).

   (a) Implement the model with one species and two substrates. Try using:
   $\tilde{\mu} = 2$,
   $k = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$,
   $Q = \begin{bmatrix} 50 \times 10^{-9} \\ 25 \times 10^{-9} \end{bmatrix}$,
   $\tilde{S} = \begin{bmatrix} 10 \\ 7.5 \end{bmatrix}$,
   $\nu = 0.5$,
   $N(0) = N_0 = 1000$,
   $S(0) = S_0 = \tilde{S}$.

   (b) Determine the R^* for both substrates.

   (c) Determine the optimal resource ratio for the population. Could you have determined this just by looking at the equations?

3.5. **(Capstone)** Using what you've learned, find values for $\tilde{S}$ so that the two species
$\tilde{\mu} = \begin{bmatrix}1.5 & 2.0\end{bmatrix}$,
$k = \begin{bmatrix}0.5 & 2.0 \\ 1.0 & 0.5\end{bmatrix}$
$Q = \begin{bmatrix}20 \times 10^{-9} & 80 \times 10^{-9} \\ 40 \times 10^{-9} & 20 \times 10^{-9}\end{bmatrix}$
will coexist.

4. Resource Ratio Theory (builds on previous)

   (a) Implement the model with three species and two substrates. Try using:
   $\tilde{\mu} = \begin{bmatrix}1 & 1.5 & 2\end{bmatrix}$,
   $k = \begin{bmatrix}0.25 & 1 & 2.25 \\ 0.75 & 1 & 0.75\end{bmatrix}$,
   $Q = \begin{bmatrix}20 \times 10^{-9} & 40 \times 10^{-9} & 60 \times 10^{-9} \\ 60 \times 10^{-9} & 40 \times 10^{-9} & 20 \times 10^{-9}\end{bmatrix}$,
   $\tilde{S} = \begin{bmatrix}10 \\ 7.5\end{bmatrix}$,
   $\nu = 0.5$,
   $N(0) = N_0 = \begin{bmatrix}1000 & 1000 & 1000\end{bmatrix}$,
   $S_0 = \tilde{S}$

   (b) Describe the succession dynamic and steady-state condition under varied substrate ratios in the source.

5. Nutrient Disturbances

   (a) Implement a model similar to the one you did in step 4.

   (b) Run the model, explain the succession dynamic and steady-state condition.

   (c) Now you are going to vary the nutrient inflow rate $\tilde{S}$. Increase or decrease the ratio of S_1 to S_2 as a periodic function.

   (d) Experiment using longer and shorter periods. What happens?

   (e) Can you find a middle value in which you see different species dominate, but none become extinct?

6. Hydraulic Disturbances

   (a) Using the model you implemented in step 5, now you are going to vary the flushing rate ν. Increase or decrease the flushing rate as a periodic function.

   (b) Experiment using longer and shorter periods. What happens?

   (c) Can you find a middle value in which you see different species dominate, but none become extinct?

There is more original research that can be done from here!

Your writeup should include:

(a) A 100-200 word abstract summarizing this assignment.

(b) An introduction. In the introduction you should:

  i. Explain the basics of the model and the importance of the problem.
  ii. Explain what you are doing for this project.

(c) Modeling and results sections for each of the steps you undertake. In these sections you should:

  i. Present the equations you are using. You do not have to fully derive them or justify them from physical principles, but please give the best explanation you can explaining where they come from.
  ii. Briefly explain how you did the numerical modeling of the equations.
  iii. Give the main results and conclusion including graphs, qualitative description of model results, answers to the questions posed in the task list and a discussion of what results you obtained.

(d) A concluding section (or paragraph) for the project

(e) A bibliography where you cite your sources of information.

Each person working on the project should provide about 5 pages of text (excluding graphs) on the project.

Grading: Each partner will hand in your report along with a group work assessment as an appendix, giving the distribution of work on this assignment between you and your partners via TurnItIn on eLearning. Submit MATLAB files electronically.

1. 5 points for the abstract.
2. 20 points for the introduction.
3. 20 points for each section of the project you work on (equations, methods, results) – include correct and complete scientific/mathematical explanation of the general problem and equations.
4. 15 points for a general conclusion to the paper
5. Each person will be graded based on the distribution of work in the group as well as the quality of the finished product.

Total: 100 points – this project is weighted at 2x the other two projects

References

[Sommer(1989)] Ulrich Sommer, editor. *Plankton Ecology: Succession in Plankton Communities.* Springer-Verlag, 1989. Chapter 3.