Homework 6

Directions: You may work with others and use outside resources, but you are subject to the rules outlined on http://www.math.tamu.edu/~jmlinhart/groupwork.html. Write neatly. Show all your work. Give at least a few sentences of explanation. Box your answer. You must staple your assignment together. Your name, UIN and section must be in the upper right hand corner of your paper. If you do not follow these instructions, your assignment will not be graded and you will receive a zero. This assignment is due **Tuesday October 15, 2013** at the beginning of your lab class.

Exercise 1. (3 points) Generalize your knowledge of how to compute the arc length of a curve in the plane to a curve in 3-dimensional space by using the arc length element

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Find the length of the *helix* that is given in parametric form by the equations

$$\left\{ \begin{array}{l} x(t) = \cos(t) \\ y(t) = \sin(t) \\ z(t) = t \end{array} \right\} \qquad \text{for } \ 0 \le t \le 6\pi$$

Exercise 2. Just the fact(orial)s! (8 points) Leonhard Euler (1707-1783) (pronounced "oiler") was one of the most prolific and influential mathematicians not only of the 18th century, but of all time. He published more than 500 books and papers *during his lifetime*, while a further 400 appeared posthumously. Project Euler is named for Leonhard Euler. Go ask Google and read a bit more about him!

One of Euler's many contributions is the *gamma function*, $\Gamma(p)$, which, when $p > 0$, can be defined by an improper integral:

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx \tag{1}$$

a) Show that $\Gamma(1) = 1$ by computing the improper integral $\displaystyle\int_0^\infty e^{-x}\, dx$.

b) Accepting for the moment that the improper integral in equation 1 does converge when $p > 0$, use integration by parts to show that $\Gamma(p+1) = p\Gamma(p)$.

c) Using the preceding two parts, deduce the numerical values of $\Gamma(2) = \Gamma(1+1)$, $\Gamma(3) = \Gamma(2+1)$, and $\Gamma(4) = \Gamma(3+1)$. Can you now see what the value of $\Gamma(n+1)$ is when n is a positive integer?

d) Prove that the improper integral in equation 1 really does converge when $p > 0$. Note that when $0 < p < 1$, the integral is improper at both ends.

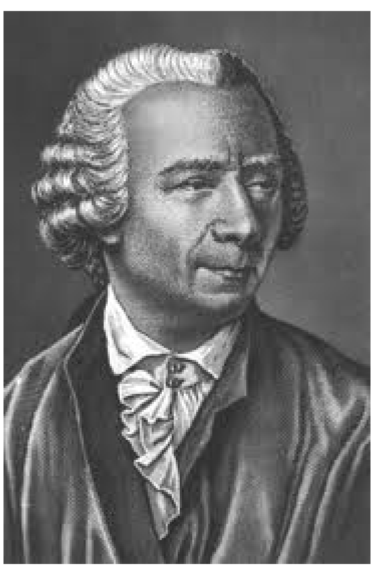
Exercise 3. (3 points) The curve given by $x = 2a\cot\theta$, $y = 2a\sin^2\theta$ is called the Witch of Agnesi. It was named for Maria Gaetana Agnesi (1718-1799), an Italian mathematician credited with writing the first book discussing both differential and integral calculus. She called the curve *versiera*, from the Italian *vertere*, "to turn". Her translator, however, had just learned Italian, and confused the word with *avversiera*, which means "wife of the devil," which is how the curve got the name Witch of Agnesi! Go ask Google for some more information about her and her curve. There is a nice animation of the Witch of Agnesi on Wolfram MathWorld.

Applying your ingenuity as to how to deal with the unknown constant a, figure out how to use MATLAB to graph this function. In MATLAB use the command `format long` to make sure you get a lot of precision in your output. Use your midpoint method routine and MATLAB's built-in `integral()` command to find its arc length of the curve from $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$. How many points do you need of your midpoint routine to get the same results as `integral()`?

Exercise 4. a) (1 point) On an earlier homework, you were asked to set up an integral for the area of a circle. Now that you know trigonometric substitution, go back and show the area is πr^2.

b) (2 point) Set up an integral for the surface area of a sphere of radius r. Box the integral. Integrate to show the sphere has surface area $4\pi r^2$. A rather miraculous simplification occurs in this integral.

Exercise 5. (3 points) In your previous homework set you showed that the region $\mathcal{R} = \{(x, y) \mid x \ge 1,\ 0 \le y \le 1/x\}$ has infinite area, but if it is rotated around the x-axis, it has finite volume. Show that the surface area is infinite. The surface is known as **Gabriel's Horn**.



Leonhard Euler

Maria Gaetana Agnesi