MATH 442: Mathematical Modeling

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Efficient Portfolio Frontier Project

Instructions You are to work alone or in groups of up to 3 to complete the following project. Groups will either be assigned by the instructor or created with the consent of the instructor. You are to use pair programming techniques and split up the to do the MATLAB tasks. You are to split the writing tasks and critique each others work, bringing everything together into one coherent report at the end. Keep track of who does what and how each person contributes to this assignment; this will also be handed in!

Your job is to explain this topic and your modeling in your own words. You will probably use references to review the topics. You should cite these references in your bibliography. If you quote from a reference, you must use quotes and a citation. If you paraphrase closely or take a problem from a reference, you use a citation. Avoid excessive quotation and paraphrasing. To accomplish this, as much as possible do not look directly at a reference while writing up your explanation, so that you use your own words. If you need to, make an outline of the major ideas and then refer back to your outline while you are writing.

Things to check before handing in your project:

1. Have you done everything you were asked to do?
2. Is the math right? If not, get it fixed.
3. Is what you are trying to explain clear?
4. Do your graphs clearly show the conclusions you reach?
5. Are the spelling, grammar, and English usage correct and concise?
6. Do you think this is A, B or C level work?
7. Run the document through a spell checker!
8. Have the Writing Center or some other good writer read your document and critique it; then edit your document taking those comments into account.

By the end of this project, you and your partners should have a thorough understanding of curve fitting, population models, and how to evaluate the models.

Investing and Overview

Every wise person saves money for retirement, rainy days, to put children through college and for other reasons. Investing money wisely so that we get a maximal return with minimal risk is clearly to our advantage, but what exactly we should invest in has no clear answer, and it is partially dependent on the personality and risk tolerance of a given investor.

Through a brokerage fund, we can buy mutual funds, stocks, bonds, and other financial products called risky assets, easily and cheaply, and thus put together a financial portfolio for ourselves. Real estate also provides numerous examples of risky assets: houses, condominiums, apartment complexes, office space, duplexes, or undeveloped land; however as these are not valued on a regular basis, they are difficult to include in the portfolio modeling we are doing here.

In this project we will calculate the efficient portfolio frontier for a given set of risky assets, and for a financial portfolio of your choice. The efficient frontier is where a given set of risky assets minimizes the risk for a given return on the investment.

Each member of the group is challenged to go out and research the type of investment products that are available. You may want to visit Morningstar, `http://www.morningstar.com/`, Vanguard `http://www.vanguard.com`, The Motley Fool, `http://www.fool.com/`, Yahoo Finance `http://finance.yahoo.com`, or other websites or books on financial investments. You may also want to consult your parents or other trusted adults/experts who have had to make investment decisions on what they have done and why they have done it.

Make a decision as to which investments you would like to make, then put together a portfolio, and calculate its efficient frontier.

The group will then compare results and see if a conclusion can be reached as to an optimal way of investing.

Along the way, you may also wish to research and learn about **black swan** events. How do these affect your investment strategy? How did your preferred portfolio fare in the most recent black swan events when the market plummeted? What do they mean by "fat tails"? You can read more about black swan investing at Gladwell [2009] and at Harrington et al. [2010].

Introduction to financial portfolios, terminology, and the efficient frontier

The efficient portfolio frontier assumes a portfolio of risky assets (stocks, bonds, mutual funds, money market accounts, etc.) and calculates how to minimize risks (portfolio variance) for a given return. We will refer to the risky asset as a stock throughout this discussion, but the discussion applies to other risky assets.

As with all financial models, the efficient frontier assumes that past performance is an indication of future results; an assumption we know to be FALSE.

First of all, how do we calculate the return on a stock? Assume we have data $(P_i, t_i)_{i=1,\cdots,n}$ of prices P_i at times t_i (the first of every month or every year). We calculate the monthly return on the investment as

$$r_i = \log\left(\frac{P_i}{P_{i-1}}\right)$$

This is the continuously compounded return on the stock (compare to continuously compounded interest on an investment). If we have dividends during month i, they can be included as

$$r_i = \log\left(\frac{P_i + D_i}{P_{i-1}}\right)$$

We can calculate the mean monthly return ($\overline{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$), and the variance of the monthly return ($\sigma^2 = \mathrm{Var}(r) = \frac{1}{n}\sum_{i=1}^{n}[r_i - \overline{r}]^2$).

If we have two or more stocks or other P_{ai} where i indicates the month and a indicates which stock, we can also calculate the covariance and correlation of the two stocks. The covariance (and the correlation which is derived from the covariance) tell us the degree to which the returns of the two stocks move together.

The covariance is given by the formula

$$\mathrm{Cov}(r_a, r_b) = \sigma_{ab} = \frac{1}{n}\sum_{i=1}^{n}(r_{ai} - \overline{r}_a)(r_{bi} - \overline{r}_b)$$

Here $\overline{r}_a = E(r_a)$ is the mean return for stock a, and n is the number of months for which we have data on our stocks.

The correlation coefficient between two stocks a and b is given by the function

$$\rho_{ab} = \frac{\mathrm{Cov}(r_a, r_b)}{\sigma_a \sigma_b}$$

In a real financial portfolio, you are likely to have more than two stocks and you will not have the same amount invested in each one. Let's say

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

represents the proportion of your financial portfolio invested in each stock, so $\sum_{a=1}^{m} w_a = 1$. We generally think $0 \leq w_A \leq 1$, but this does not need to be true in an actual portfolio that allows borrowing and short sales. We can use $\mathbf{w}$ to calculate the mean, variance and standard deviation of the portfolio.

We use $E(r_p)$ to denote the mean return of the entire portfolio, whereas $E(\mathbf{r}_p)$ is a vector of means for each stock in the portfolio.

$$\text{Mean return for entire portfolio:} \qquad E(r_p) = \overline{r}_p = \sum_{a=1}^{m} w_a \overline{r}_a$$

$$\text{Variance for entire portfolio:} \qquad \mathrm{Var}(r_p) = \sigma_{r_p}^2 = \sum_{a=1}^{m} w_a w_b \sigma_{ab}$$

This is more compactly expressed using matrix notation. Let S be the variance-covariance matrix for the portfolio. The indices run over the risky assets in the portfolio.

$$S = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \cdots & \sigma_{2m} \\ \vdots & & & & \\ \sigma_{m1} & \sigma_{m2} & \sigma_{m3} & \cdots & \sigma_{mm} \end{bmatrix}$$

then the portfolio variance is given by

$$\mathrm{Var}(r_p) = \mathbf{w}^T S \mathbf{w}$$

Where $\mathbf{w}$ gave us the amount invested in each stock.

If

$$E(\mathbf{r}_p) = \overline{\mathbf{r}}_p = [E(r_1), E(r_2), \ldots E(r_m)]^T$$

is the mean return vector for each stock in the portfolio, then the expected return for the entire portfolio is

$$E(r_p) = \mathbf{w}^T \overline{\mathbf{r}}$$

You can plot the mean return for the portfolio versus the portfolio standard deviation (the square root of the variance).

All possible portfolios are the feasible set. Those portfolios giving the highest return for a given variance are on the **Efficient Frontier** which is what we want to model.

In mathematical terms what we want to do is for a given return μ and a given set of stocks in a portfolio, we want to find $\mathbf{w}$ so that

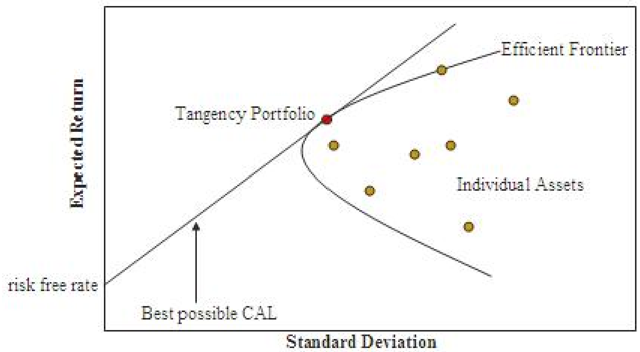
$$\min \sum_a \sum_b w_a w_b \sigma_{ab} = \mathrm{Var}(r_p)$$

subject to

$$\begin{aligned} \sum_a w_a r_a &= \mu = E(r_p) \\ \sum_a w_a &= 1 \end{aligned}$$

Notice that there are hidden matrix computations here!

The efficient frontier is illustrated in the following diagram from Wikipedia [2004]



In this diagram, the feasible portfolios are to the right of the curved (hyperbolic) line. The efficient frontier is the top half of the hyperbola. The hyperbola itself is called the envelope of the feasible region.

Calculating the Efficient Portfolio Frontier

Method 1

Here I follow the work of Merton [1972]. This article is available to you on eLearning as **EfficientPortfolioFrontier** – the material you should pay the most attention to is in section II.

Merton notes that the minimization problem can be solved by the method of Lagrange multipliers. If we wish to minimize $F(\mathbf{x})$ with constraints $g(\mathbf{x}) = g_0$ and $h(\mathbf{x}) = h_0$, we use Lagrange multipliers λ_1 and λ_2 and instead minimize $F(\mathbf{x}) - \lambda_1(g(\mathbf{x}) - g_0) - \lambda_2(h(\mathbf{x}) - h_0)$ with respect to $\mathbf{x}$, λ_1 and λ_2. We take the partial derivatives with respect to each component of $\mathbf{x}$ as well as the partial derivatives with respect to λ_1 and λ_2 and set all equal to zero. The partial derivatives with respect to λ_1 and λ_2 give us our constraints back.

In the efficient portfolio frontier problem, if we know that all stocks are risky ($\sigma_a^2 > 0$) and that no stock is a linear combination of the rest, we also know that there is a unique minimum.

Our minimization problem is

$$\min\left(\mathrm{Var}(r_p)\right)$$

subject to

$$\begin{aligned}
\mathrm{Var}(r_p) &= \sum_{a=1}^{m}\sum_{b=1}^{m} w_a w_b \sigma_{ab} \\
E &= \mu = \sum_{a=1}^{m} w_a \overline{r}_a \\
1 &= \sum_{a=1}^{m} w_a
\end{aligned}$$

The method of Lagrange multipliers is applied; we consider

$$F(w_1, \dots, w_m, \lambda_1, \lambda_2) = \left(\sum_{a=1}^{m} \sum_{b=1}^{m} w_a w_b \sigma_{ab} \right) - \lambda_1 \left[\left(\sum_{a=1}^{m} w_a \overline{r}_a \right) - \mu \right] - \lambda_2 \left[\left(\sum_{a=1}^{m} w_a \right) - 1 \right]$$

To do this we set

$$\frac{\partial F}{\partial w_a} = 0 \quad \text{for } a = 1, \dots m, \quad \text{and} \quad \frac{\partial F}{\partial \lambda_i} = 0, \quad \text{for } i = 1,\ 2$$

and solve for $\{w_a\}$, λ_1 and λ_2.

Without going into all of the algebra involved, this solution is obtained by defining scalars A, B and C by

$$\begin{aligned} A &= \sum_{k=1}^{m} \sum_{j=1}^{m} S_{kj}^{-1} \overline{r}_j \\ B &= \sum_{k=1}^{m} \sum_{j=1}^{m} S_{kj}^{-1} \overline{r}_j \overline{r}_k \\ C &= \sum_{k=1}^{m} \sum_{j=1}^{m} S_{kj}^{-1} \end{aligned}$$

recalling that $S = [\sigma_{ab}]$ is the variance/covariance matrix. Then

$$\begin{aligned} \lambda_1 &= \frac{CE - A}{D} \\ \lambda_2 &= \frac{B - AE}{D} \\ \text{where} \quad D &= BC - A^2 > 0 \end{aligned}$$

This in turn gives the weight of each stock as

$$w_a = \frac{E \sum_{j=1}^{m} S_{kj}^{-1} (C\overline{r}_j - A) + \sum_{j=1}^{m} S_k^{-1} j (B - A\overline{r}_j)}{D}, \qquad a = 1, 2, \dots, m$$

and we can find the variance as a function of the return as

$$\text{Var}(r_p) = \frac{CE^2 - 2AE + B}{D}$$

The efficient frontier in mean-variance space is a parabola, in mean-standard deviation space, we get one branch of a hyperbola.

Method 2

This section follows Benninga [2000].

If you have any two portfolios $\mathbf{w}$ and $\mathbf{v}$ that are on the envelope, you can obtain the rest by taking convex combinations of the two:

$$\mathbf{x} = \alpha\mathbf{w} + (1-\alpha)\mathbf{v}$$

is also on the envelope. The efficient frontier is the top half of the envelope.

To find a portfolio that is on the envelope, let c be a constant. Then we use the notation $\mathbf{r} - c$ to indicate the column vector

$$\mathbf{r} - c = \begin{bmatrix} E(r_1) - c \\ E(r_2) - c \\ \vdots \\ E(r_m) - c \end{bmatrix}$$

Let the vector $\mathbf{z}$ solve the system of simultaneous linear equations $\mathbf{r} - c = S\mathbf{z}$. This produces a portfolio $\mathbf{w}$ on the envelope of the feasible set by

$$\begin{aligned} \mathbf{z} &= S^{-1}(\mathbf{r} - c) \\ \mathbf{w} &= \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \\ w_i &= \frac{z_i}{\sum_{A=1}^{m} z_A} \end{aligned}$$

Choosing two different values of c and then taking all of the convex combinations of the two portfolios will give you the envelope of the feasible region; the top half of which is the efficient frontier.

Your modeling goals are:

I. Find the efficient portfolio frontier for the given portfolio.

II. Each partner is to choose a portfolio of his or her own, giving the reasoning for your choice of risky assets to include in your portfolio.

III. Find the efficient portfolio frontier for this portfolio.

IV. Discuss black swan events and how to invest given the inevitability of black swan events.

V. Notice that the variance takes account of upturns and downturns in the price of a risky asset, but we are only adversely affected by the downturns. If you only wanted to consider the downturns to be part of the risk, what would you calculate? What steps might you take to solve the minimization problem of finding this minimal risk for a given return?

Your write-up should be paper/report roughly 1500-3000 words long, not including the bibliography. It should include:

(a) A 100-200 word abstract summarizing your paper.

(b) An introduction. In the introduction you should explain the minimization problem for the Efficient Portfolio Frontier using the method of Lagrange Multipliers.

(c) A modeling and methods section in which you explain the efficient portfolio frontier model, and how you compare the results from different portfolios. (i.e. how you determine which portfolio is best)

(d) A section for each portfolio modeled, including an explanation of which risky assets were chosen and why, and the efficient frontier results, including a graph of the efficient frontier.

(e) A section on black swan events and how one might invest with these in mind.

(f) A section on how you might change the problem to account for the fact that only price downturns on our risky assets are truly bad.

(g) An overall results and conclusion section in which you include

  i. An overall summary of your results.
  ii. An evaluation of which portfolios were the best.
  iii. Any criticisms of the model that you can see using your "uncommon sense".
  iv. A concluding paragraph that sums up the paper.

(h) A bibliography where you cite your sources of information.

Grading Each individual will hand in a copy of your report along with their group work assessment. Submit the MATLAB and LaTeX files on eLearning. You will use TurnItIn on eLearning to submit the PDF of your final report. Include the group work assessment as an appendix to your report.

1. 5 points for the abstract.

2. 10 points for an introduction including correct and complete scientific/mathematical explanation of the efficient portfolio frontier equations and assumptions, and how to find financial data.

3. 10 points for a good explanation of the methods you use to fit the models and determine which are performing the best.

4. 20 points for each sections on a portfolios modeled (assuming 3 of these).

5. 15 points for your overall results and conclusions.

6. Each person will be graded based on the distribution of work in the group as well as the quality of the finished product.

Total: 100 points; weight is 2x that of previous projects or 100 points.

References

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Malcolm Gladwell. *What the Dog Saw.* Little, Brown and Company, 2009. URL `http://www.gladwell.com/pdf/blowingup.pdf`. available at `http://www.gladwell.com/pdf/blowingup.pdf`, originally published in *The New Yorker.*

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Robert Merton. An analytic derivation of the efficient portfolio frontier. *The Journal of Financial and Quantitative Analysis*, 7(4):1851–1872, 1972.

Dessislava A. Pachamanova and Frank J. Fabozzi. *Simulation and Optimization in Finance.* John Wiley and Sons, 2010. Pages 245-276.

Wikipedia. Efficient frontier — Wikipedia, the free encyclopedia, 2004. URL `\url{http://en.wikipedia.org/wiki/Efficient_frontier}`. [Online; accessed 21-March-2012].