Mathematical Modeling of a Zombie Outbreak

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Zombies!

- Zombies are the walking undead. Slow moving ex-human horrors, they feast on human flesh and turn humans into zombies.
- According to MSNBC, Zombies are worth $5 billion in today's economy. (Ogg [2011])
- Humans vs. Zombies (HvZ) is played on college campuses throughout the US (at TAMU since 2009) and on six continents. (HvZauthors [2012])
- Even the CDC has emergency prepardness plans for the Zombie Apocalypse. (CDCAuthors [2011])

Predator-Prey Hypotheses

Hypotheses:

- Zombies, Z, are walking undead; feed on human flesh
- Humans, S (for susceptibles), are prey
- Zombies kill or zombify humans via mass-action interaction
- Zombies bite humans to create more zombies (mass-action)
- Zombies are subject to decay

Predator-Prey Differential equations

Lotka-Volterra predator-prey equations (Lotka [1910], Volterra [1931]):

$$\begin{aligned}\frac{dS}{dt} &= \pi S - \beta SZ \\ \frac{dZ}{dt} &= \alpha SZ - \delta Z\end{aligned}$$

These are **autonomous** differential equations, they do not depend on t, only on S and Z.

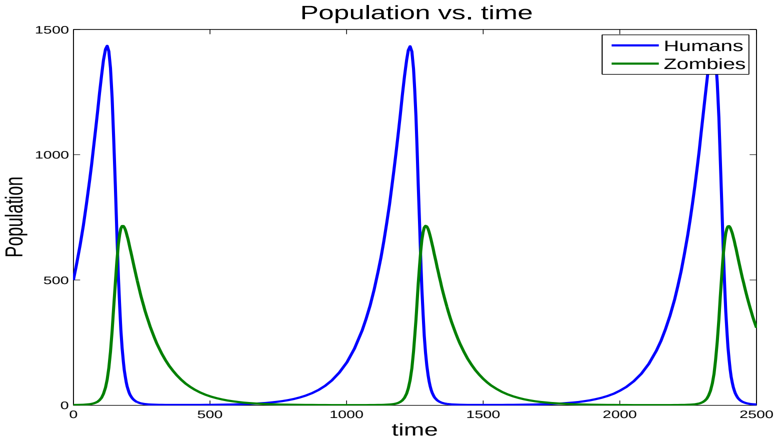
Seemingly simple, they have no analytic solution, and can only be approximated numerically.

Human and Zombie populations over time.

Parameters:

$\pi = 0.01, \quad \beta = 10^{-4}, \quad \alpha = 5 \times 10^{-5}, \quad \delta = 0.01$

Initial conditions: $S_0 = 500$ and $Z_0 = 1$



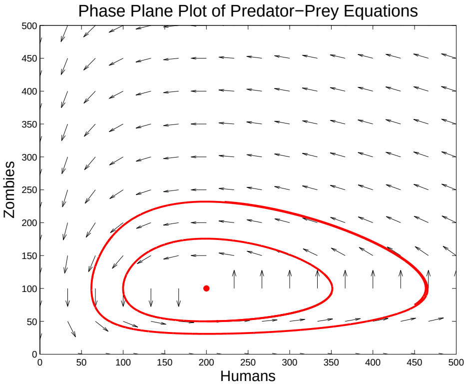
Phase plane for the predator-prey equations

Using the chain rule:

$$\frac{dZ}{dt}\frac{dt}{dS} = \frac{dZ}{dS}$$

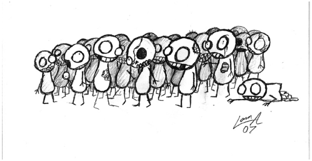
Parameters:

$\pi = 0.01, \quad \beta = 10^{-4}, \quad \alpha = 5 \times 10^{-5}, \quad \delta = 0.01$



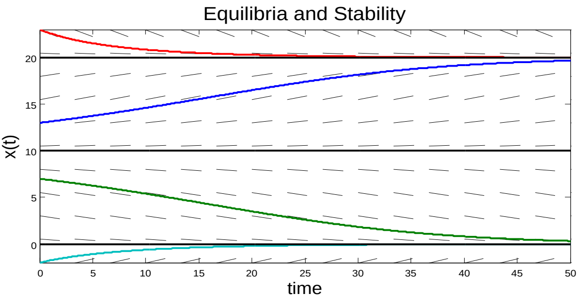
Questions!

- Aside from plotting a phaseplane how can I tell how changing the initial conditions will affect the solution to a system of differential equations?
- Does this type of model always give this type of cyclic behavior?

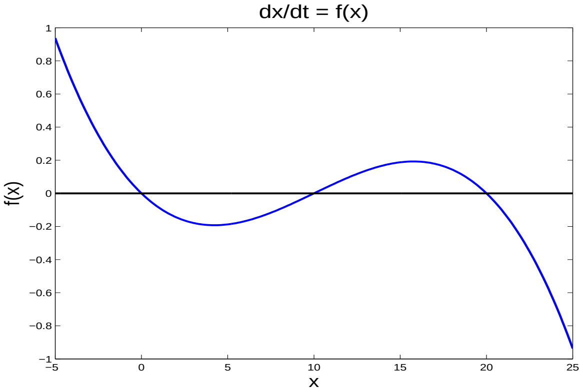


Changes in initial conditions: Equilibria and Stability

- An equilibrium solution $x(t) = x_0$ for an autonomous differential equation $x' = f(x)$ occurs when $f(x_0) = 0$.
- Example: $x' = 0.1x\left(\frac{x}{10} - 1\right)\left(1 - \frac{x}{20}\right)$
- What is a stable equilibrium? Change the initial conditions a little bit, and the model goes to the equilibrium in the long-term.



Changes in initial conditions: Equilibria and Stability (cont.)



Stable when $f'(x_0) < 0$.

Equilibria and stability in higher-dimensional systems

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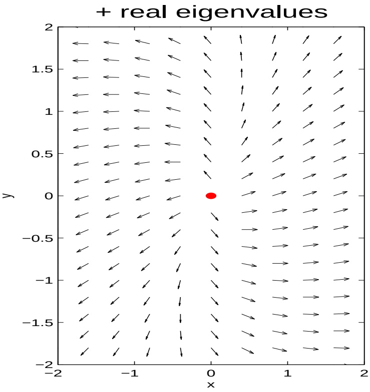
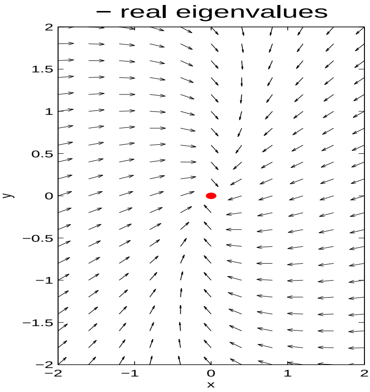
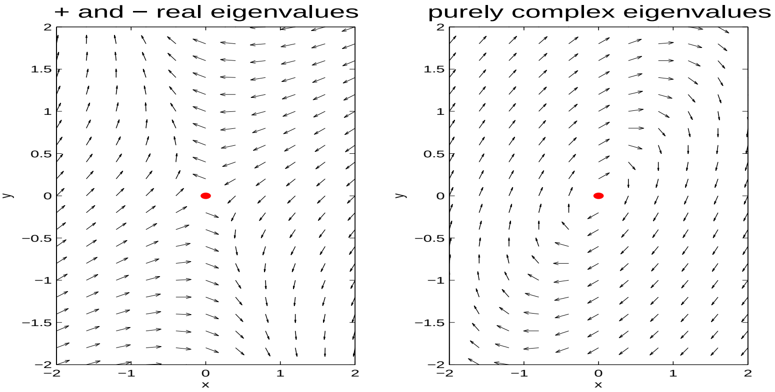
Simplest case: $\vec{x}' = A\vec{x}$ where $\vec{x}$ is a 2-d column vector, A is a 2×2 matrix, we find the eigenvalues of A:





Equilibria and stability in higher dimensional systems (cont.)

$\vec{x}' = A\vec{x}$ where $\vec{x}$ is a 2-d column vector, A is a 2×2 matrix, we find the eigenvalues of A:



Equilibria and stability in higher dimensional systems (cont.)

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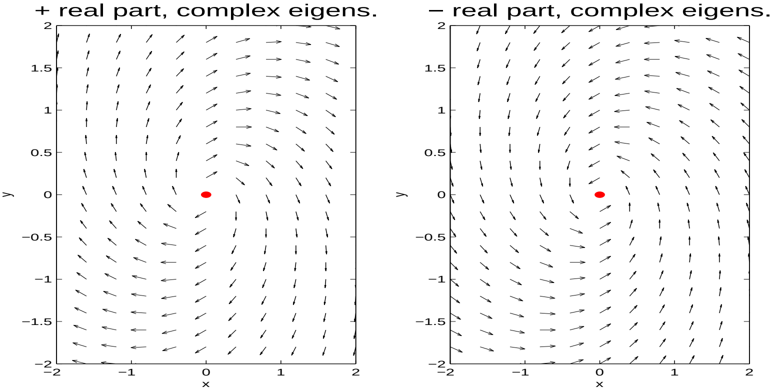


The bottom line: negative real eigenvalues or negative real parts means stability.

Stability for nonlinear autonomous systems

Nonlinear autonomous system $\vec{x}' = \vec{F}(\vec{x})$; $\quad \vec{F}(\vec{x}_0) = \vec{0}$

Find the eigenvalues of the Jacobian matrix:

$$J(\vec{x}_0) = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1}(\vec{x}_0) & \dfrac{\partial F_1}{\partial x_2}(\vec{x}_0) \\ \\ \dfrac{\partial F_2}{\partial x_1}(\vec{x}_0) & \dfrac{\partial F_2}{\partial x_2}(\vec{x}_0) \end{bmatrix}$$

If all real parts are negative, we have hyperbolic equilibria and stability.

If we add zero eigenvalues, we must analyze the equations to determine stability.

Analysis of the Predator-Prey model

Equilibria:

$$\frac{dS}{dt} = \pi S - \beta SZ = 0 \implies S = 0 \text{ or } Z = \frac{\pi}{\beta}$$

$$\frac{dZ}{dt} = \alpha SZ - \delta Z = 0 \implies Z = 0 \text{ or } S = \frac{\delta}{\alpha}$$

We have two equilibria: $(0,0)$ and $\left(\frac{\delta}{\alpha}, \frac{\pi}{\beta}\right)$

Jacobian matrix:

$$J(S,Z) = \begin{bmatrix} \pi - \beta Z & -\beta S \\ \alpha Z & \alpha S - \delta \end{bmatrix}$$

Jacobian matrices at equilibria:

$$J(0,0) = \begin{bmatrix} \pi & 0 \\ 0 & -\delta \end{bmatrix}$$

$$J\left(\frac{\delta}{\alpha}, \frac{\pi}{\beta}\right) == \begin{bmatrix} 0 & -\frac{\beta\delta}{\alpha} \\ \frac{\alpha\pi}{\beta} & 0 \end{bmatrix}$$

Analysis of Predator-Prey model (cont.)

At $(0, 0)$

$$\lambda = \pi \text{ or } \lambda = -\delta$$

This is a saddle point; it is not really of interest to us.

At $\left(\frac{\delta}{\alpha}, \frac{\pi}{\beta}\right)$,

$$\lambda = \pm i\sqrt{\delta\pi}$$

we get periodic circulation.

Since this result is independent of choice of the parameters α, β, π, δ, we know this behavior is generic.

Now, can anyone think of any criticisms of this model?

Criticisms of the Predator-Prey model

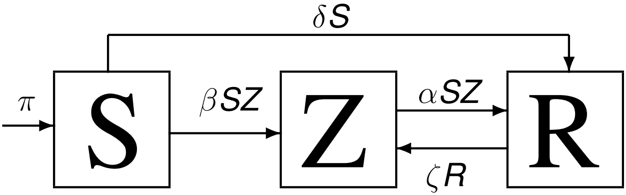
- Humans are not prey. We fight back.
- We need to look at another model!

Zombie disease models were popularized in 2009 with the publication of "When Zombies Attack" in *Infectious Disease Modelling Research Progress*. (Munz et al. [2009])

They work with three classes:

- Susceptibles, S, or uninfected humans.
- Zombies, Z, the walking undead (normally I for infected)
- Removed, R, the dead (normally recovered).

Basic Zombie Model



$$\frac{dS}{dt} = \pi - \beta SZ - \delta S$$

$$\frac{dZ}{dt} = \beta SZ - \alpha SZ + \zeta R$$

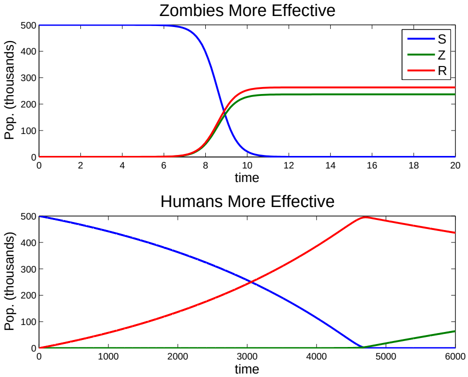
$$\frac{dR}{dt} = \alpha SZ - \zeta R$$

Model Results: doomsday

Parameter values $\pi = 0.0001, \quad \zeta = 0.0001, \quad \delta = 0.0001$

Both started with all humans, no zombies.

First plot $\alpha = 0.005 < \beta = 0.0095$; zombies more effective
Second plot $\alpha = 0.01 > \beta = 0.008$; humans more effective



Equilibria of the basic Zombie model

As originally written, the equations don't permit an equilibrium because the birthrate π is a constant. We will consider a short time-frame where birth and death rates are set to zero. $\pi = \delta = 0$

$$\begin{aligned} \frac{dS}{dt} &= -\beta S Z = 0 \\ \frac{dZ}{dt} &= \beta S Z - \alpha S Z + \zeta R = 0 \\ \frac{dR}{dt} &= \alpha S Z - \zeta R = 0 \end{aligned}$$

Two equilibria: $(N, 0, 0)$ (all humans) and $(0, N, 0)$ (all zombies; doomsday).

The system of equations and Jacobian matrix for the system

$$\begin{aligned}
\frac{dS}{dt} &= -\beta SZ \\
\frac{dZ}{dt} &= \beta SZ - \alpha SZ + \zeta R \\
\frac{dR}{dt} &= \alpha SZ - \zeta R
\end{aligned}$$

$$J = \begin{bmatrix} -\beta Z & -\beta S & 0 \\ \beta Z - \alpha Z & \beta S - \alpha S & \zeta \\ \alpha Z & \alpha S & -\zeta \end{bmatrix}$$

Stability for the zombie-free equilibrium

The eigenvalues of the Jacobian matrix at $(N, 0, 0)$:

$$\lambda = 0, \quad \frac{-(\zeta + [\alpha - \beta]N) \pm \sqrt{(\zeta + [\alpha - \beta]N)^2 + 4\beta\zeta N}}{2}$$

This will have a root for λ that is positive, and therefore the zombie-free equilibrium is not stable.

This is generic behavior; it does not depend on parameter values aside from that they are positive and $\alpha > \beta$.

Doomsday equilibrium

Eigenvalues of the Jacobian matrix at $(0, N, 0)$:

$$\lambda(\lambda + \beta N)(\lambda + \zeta) = 0$$

$$\lambda = 0, \quad \lambda = -\beta N, \qquad \lambda = -\zeta$$

A zero eigenvalue does not guarantee stability in general.

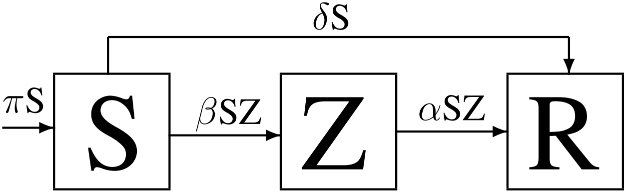
Looking at the ODEs, it is stable, and the behavior is generic, since it doesn't depend on the values of the parameters.

What criticisms would you make of this zombie model?

Criticisms of this zombie model.

- Allowing the dead to spontaneously come back as zombies rigs the game against us. (All my students)
- A corpse cremated to ash should not be able to come back as a zombie. (All)
- If zombies are animated corpses, shouldn't they be subject to decay? (Cho, Hughes, Marek)
- If zombie-ism is a disease, then dead should mean **dead**. (Many)
- Constant birthrate is not used in modern population modeling. πS (Malthusian model) is more common. (All)

Basic Disease model

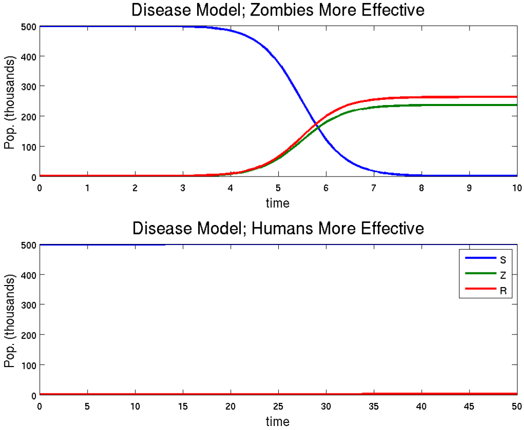


$$
\begin{aligned}
\frac{dS}{dt} &= \pi S - \beta SZ - \delta S \\
\frac{dZ}{dt} &= \beta SZ - \alpha SZ \\
\frac{dR}{dt} &= \delta S + \alpha SZ
\end{aligned}
$$

The dead cannot become zombies; dead is dead. Zombies, however, are immortal unless beheaded or cremated by humans.

Disease results

Parameter values $\pi = 0.0001, \quad \delta = 0.0001$
First plot $\alpha = 0.005 < \beta = 0.0095$; zombies more effective
Second plot $\alpha = 0.01 > \beta = 0.008$; humans more effective



Stability of the disease model results

$\pi = \delta = 0$, omit removed (dead) class since it doesn't influence the others.

$$\begin{aligned} \frac{dS}{dt} &= -\beta SZ = 0 \\ \frac{dZ}{dt} &= \beta SZ - \alpha SZ = (\beta - \alpha)SZ = 0 \end{aligned}$$

Equilibria: $(S, 0)$ and $(0, Z)$

$$J(S, Z) = \begin{bmatrix} -\beta Z & -\beta S \\ (\beta - \alpha)Z & (\beta - \alpha)S \end{bmatrix}$$

Stable equilibria for both doomsday and human survival

Humans win, eigenvalues $(N, 0)$, $\alpha > \beta$:

$$\lambda = \frac{-(\alpha - \beta)N \pm \sqrt{(\alpha - \beta)^2 N^2 - \beta N}}{2}$$

Zombies win, eigenvalues $(0, Z)$, $\beta > \alpha$:

$$\lambda = \frac{-\beta Z \pm \sqrt{\beta^2 Z^2 + 4(\alpha - \beta)Z}}{2}$$

In both cases, eigenvalues are always negative. These are both stable equilibrium.

Student work

Fred Doe and Amanda Roehling (Doe and Roehling [2012]):

- Arming the populace: armed citizen $5\times$ as effective as unarmed.
- Result: if done quickly enough and humans effective enough, humans survive.
- If the populace cannot handle weapons, then use military intervention: zombies were more effective than humans; military troops more effective than zombies.
- Results: depended on how many troops and when deployed. A strong, fast, response was most likely to be effective.

Sphere of influence in a supernatural zombie model

Frances Withrow and Tyler Gleasman (Withrow and Gleasman [2012]):

- It is not possible for humans or zombies to interact with everyone in one time step. Mass action becomes

$$\alpha \frac{S}{1+\kappa S} Z$$

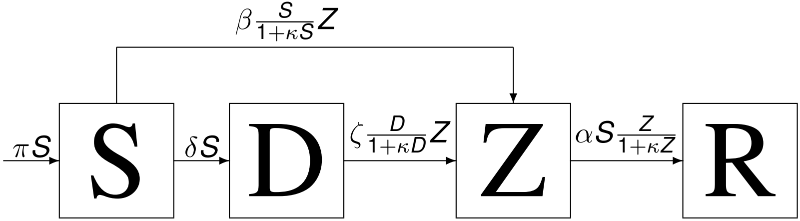
  Observe

$$\lim_{S\to\infty} \frac{S}{1+\kappa S} = \frac{1}{\kappa} \qquad \lim_{S\to 0} \frac{S}{1+\kappa S} = 1$$

  $1/\kappa$ is the maximum number of susceptibles a zombie can interact with in a time step.
- If the unit of time is a day, $\kappa \approx 1/500$ seems like a reasonable value.

Sphere of influence model, cont.

- Supernatural zombies. Zombies can bring corpses back to life.
- Zombies can be beheaded or cremated and made permanently dead.



Sphere of influence ODEs

$$
\begin{aligned}
\frac{dS}{dt} &= \pi S - \frac{\beta SZ}{1+\kappa S} - \delta S \\
\frac{dZ}{dt} &= \frac{\beta SZ}{1+\kappa S} - \frac{\alpha SZ}{1+\kappa Z} + \frac{\zeta DZ}{1+\kappa D} \\
\frac{dR}{dt} &= \frac{\alpha SZ}{1+\kappa Z} \\
\frac{dD}{dt} &= \delta S - \frac{\zeta DZ}{1+\kappa D}
\end{aligned}
$$

Sphere of influence results

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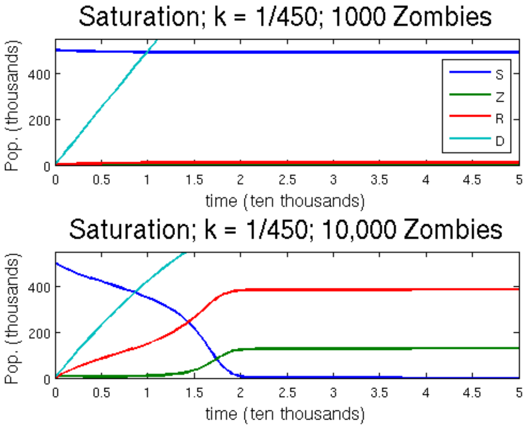
Start with 500,000 humans.



Sphere of influence stability

Short time scale $\pi = \delta = 0$. I omit the removed class, since it doesn't influence any of the other classes.

The zombie-free equilibrium has Jacobian:

$$J(S, 0, D) = \begin{bmatrix} 0 & -\dfrac{\beta S}{1+\kappa S} & 0 \\ 0 & \dfrac{\beta S}{1+\kappa S} - \alpha S + \dfrac{\zeta D}{1+\kappa D} & 0 \\ 0 & -\dfrac{\zeta D}{1+\kappa D} & 0 \end{bmatrix}$$

Eigenvalues are given by

$$\lambda = 0 \quad \text{or} \quad \lambda = \frac{\beta S}{1+\kappa S} - \alpha S + \frac{\zeta D}{1+\kappa D}$$

The latter is negative since $-\alpha S$ dominates.

I argue this is stable despite the zero eigenvalues, and generic behavior.

Sphere of influence stability

The doomsday equilibrium has Jacobian (no dead because the dead eventually all become zombies):

$$J(0, Z, 0) = \begin{bmatrix} -\beta Z & 0 & 0 \\ \beta Z - \dfrac{\alpha Z}{1+\kappa Z} & 0 & \zeta Z \\ 0 & 0 & -\zeta Z \end{bmatrix}$$

Eigenvalues are given by $\lambda = 0$, $\lambda = -\beta Z$, $\lambda = -\zeta Z$.

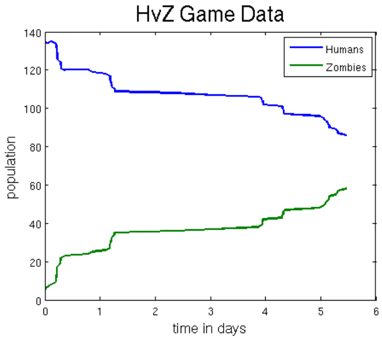
This is stable and generic behavior

Other directions

- Inwhi (Andy) Cho, Aron Hughes, Michael Marek: zombie decay
- Maria Troyanova-Wood: two-climate model with migration
- Stochastic models
- Agent-based models

Zombie meta-study with Humans vs. Zombies data from multiple campuses.

2011 Texas A&M HvZ data:



Open questions

- Is it possible for humans and zombies to coexist? Can we find a stable coexistence equilibrium?
- Can we find a model with a stable disease-free equilibrium and an unstable doomsday equilibrium?
- How should we model the HvZ game?
- What can HvZ tell us about zombie models?

Wrap-up

- To prepare for a zombie outbreak, the best tactic we've identified is to get people trained with weapons.
- Keep the military prepared and teach children how to shoot and handle weapons from an early age.
- To protect your family: weapons, food, water, a remote hide-away.
- Keep those math students busy trying to improve the models and save us all.

Thanks for coming and thanks for listening!

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